A Class of Monotone Operator Functions Related to Electrical Network Theory

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ABSTRACT

A new class of monotone functions which map positive operators to positive operators is defined and studied. The class is motivated by electrical network theory. Various properties of these functions are given, including behavior under composition and an electrically motivated duality theory.

1. INTRODUCTION

In this paper we define and study a class of monotone functions which map positive operators on one finite dimensional space to positive operators on another space. The particular class under consideration is motivated by the study of electrical networks.

In most studies of monotone operator functions, the functions are scalar functions extended to operators using the operational calculus furnished by the spectral theorem [13, 17]. Our functions do not fit into this pattern. In fact, all of our functions satisfy the equation $\phi(\alpha A)=\alpha\phi(A)$, where α is a scalar; thus the only scalar function available is the constant multiple. We will call our functions *posotone functions.*

In Sec. 2 we give some preliminary results. The definition of posotone functions and a discussion of their behavior under composition is given in Sec. 3. In the next two sections the concept of duality and the relationship of posotone functions to electrical network theory are considered. Extensions of this work are considered in the last section.

2. PRELIMINARIES

Let V and W denote finite dimensional complex vector spaces, and let $\mathfrak{B}(V)$ and $\mathfrak{B}(W)$ denote the algebras of all linear operators on V and W respectively. For a linear operator $A \in \mathfrak{B}(V)$, let A^* denote the adjoint operator defined by $(Ax,y)=(x,A^*y)$ for all vectors $x,y \in V$. If $A=A^*$, then A is said to be *Hermitian*. A *positive operator* is a Hermitian operator $A \in \mathfrak{B}(V)$ such that $(Ax,x) \geqslant 0$ for all $x \in V$; let $\mathfrak{P}(V)$ denote the cone of all positive operators on V. For Hermitian operators A and B, we write $A \geqslant B$ if $A-B$ is a positive operator.

A linear function $L: \mathfrak{B}(V) \to \mathfrak{B}(W)$ is called *positive* if $L(A)$ is a positive operator whenever A is positive. A linear function L is called *completely positive* if there are linear operators $Z_i : V \to W$, $i=1,\dots,k$ such that for all operators $A \in \mathfrak{B}(V)$,

$$L(A)=\sum_{i=1}^{k} Z_i A Z_i^*. \tag{1}$$

Alternatively, we may let $V^{(k)}$ denote the direct sum of k copies of V, and $A^{(k)}$ the direct sum of k copies of A. Then (1) may be rewritten

$$L(A)=ZA^{(k)}Z^*, \tag{1$'$}$$

where Z is the appropriate operator $Z: V^{(k)} \to W$. In terms of matrices, $Z=[Z_1 \quad Z_2 \quad \cdots \quad Z_k]$ and

$$A^{(k)}=\begin{bmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{bmatrix}.$$

Given the linear function $L: \mathfrak{B}(V) \to \mathfrak{B}(W)$, we define the linear function $L^{(2)}: \mathfrak{B}(V^{(2)}) \to \mathfrak{B}(W^{(2)})$ by

$$L^{(2)}\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\begin{bmatrix} L(A) & L(B) \\ L(C) & L(D) \end{bmatrix}. \tag{2}$$

If L is completely positive, so that $L(A)=\sum_{i=1}^{k} Z_i A Z_i^*$, then we may easily see that

$$L^{(2)}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \sum_{i=1}^{k} \begin{bmatrix} Z_i & 0 \\ 0 & Z_i \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Z_i & 0 \\ 0 & Z_i \end{bmatrix}^*.$$

That is, if L is completely positive, so is $L^{(2)}$. In an analogous manner one may define $L^{(k)}$; then if L is completely positive, so is $L^{(k)}$.

Using this latter observation, we may give an example of a positive linear function which is not completely positive. Let V and W be two dimensional complex spaces, and let $L(A)=A^t$. That is,

$$L\begin{bmatrix} a & b \\ b^* & c \end{bmatrix} = \begin{bmatrix} a & b^* \\ b & c \end{bmatrix}. \tag{3}$$

Then L is clearly a positive linear function. Now define

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix};$$

then

$$L^{(2)}(M) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix M is positive, but $L^{(2)}M$ is not, and therefore L is not completely positive.

Completely positive linear functions were introduced by Stinespring [21] and extensively studied by Arveson [7]. They defined the linear function L to be completely positive if the function $L^{(k)}$ was positive for all k. We have seen above that functions which are completely positive in our sense also satisfy the definition of Stinespring; the converse has been proved by Choi

[11]. The above example of a positive function which is not completely positive is given by Arveson; another has been given by Bose and Mitra [8]. Stormer has made a study of positive linear maps which are not necessarily completely positive [22].

The nonlinear character of our posotone functions will be due to the use of the *shorted operator*. This concept, also called the *generalized Schur complement* [10, 12] has widespread applications in electrical network theory [4], statistics [1] and differential operator theory [16].

Let S be a subspace of W, and let A be a positive linear operator on W with partitioned matrix

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{4}$$

where $A_{11}: S \to S$, $A_{12}: S^{\perp} \to S$, etc. The *shorted operator of* A, $\mathcal{S}(A)$, is defined as an operator on S by

$$\mathcal{S}(A)=A_{11}-A_{12}A_{22}^{+}A_{21}. \tag{5}$$

In the definition A_{22}^{+} represents the Moore-Penrose generalized inverse of A_{22}. [To construct A_{22}^{+}, observe that since A_{22} is positive, it may be written as the orthogonal direct sum $A_{22}=\alpha\oplus 0$, where α is an invertible operator on $\operatorname{ran}(A_{22})$. Then $A_{22}^{+}=\alpha^{-1}\oplus 0$.]

Where explicit mention of the subspace is necessary, we will let $\mathcal{T}(A)$ denote a short to the subspace T, etc. More commonly, A will be given as a 2×2 partitioned matrix as in (4); then $\mathcal{S}(A)$ is defined by (5).

Related to the concept of the shorted operator is the concept of the parallel sum. If A and B are positive operators on W, the parallel sum $A:B$ is defined by

$$A:B=A(A+B)^{+}B. \tag{6}$$

This definition of the parallel sum is directly motivated by electrical network theory [3, 4]. If A and B are invertible, an equivalent definition is $A:B=(A^{-1}+B^{-1})^{-1}$, which might suggest the generalization $(A^{+}+B^{+})^{+}$ instead of (6). This latter definition is not correct in the electrical situation, but it has other uses [20]. For an example in which $A:B\neq (A^{+}+B^{+})^{+}$, let $A=I$ and $B=0$.

Various important properties of the shorted operator and parallel addition are summarized in the following two lemmas, which are proved in [2] and [3]. Alternative proofs, giving the appropriate infinite dimensional generalizations, are given in [5].

LEMMA 1 *Let V and W be finite dimensional complex vector spaces. Let A, B and C be positive operators on W, and let $M: W \to V$ be a linear operator. Let S and T be subspaces of W, and let $R = S \cap T$. Let α be a non-negative real number. Then*

$$A:B = B:A, \tag{7a}$$

$$A:(B:C) = (A:B):C, \tag{7b}$$

$$(\alpha A):(\alpha B) = \alpha(A:B), \tag{7c}$$

$$M(A:B)M^* \leqslant (MAM^*):(MBM^*), \tag{7d}$$

$$\mathfrak{S}(\alpha A) = \alpha \mathfrak{S}(A), \tag{7e}$$

$$\mathfrak{S}(\mathfrak{T}(A)) = \mathfrak{T}(\mathfrak{S}(A)) = \mathfrak{R}(A), \tag{7f}$$

$$\mathfrak{S}(A:B) = \mathfrak{S}(A):\mathfrak{S}(B), \tag{7g}$$

$$\mathfrak{S}(A+B) \geqslant \mathfrak{S}(A) + \mathfrak{S}(B), \tag{7h}$$

with equality if and only if

$$[\operatorname{ran}(A - \mathfrak{S}(A)) + \operatorname{ran}(B - \mathfrak{S}(B))] \cap S = 0.$$

LEMMA 2 *Let W be a finite dimensional complex vector space, S a subspace of W. Let A be a positive operator on W, and let $\{E_n\}$ be a sequence of positive operators on W decreasing monotonically to 0. Then $\mathfrak{S}(A + E_n)$ decreases monotonically to $\mathfrak{S}(A)$.*

LEMMA 3. *Let L be a completely positive linear function, and let A and B be positive operators. Then $L(A:B) \leqslant L(A):L(B)$.*

Proof. Observe that $(A:B)^{(k)} = A^{(k)}:B^{(k)}$. Then using (7d), $L(A:B) = Z(A:B)^{(k)}Z^* = Z(A^{(k)}:B^{(k)})Z^* \leqslant (ZA^{(k)}Z^*):(ZB^{(k)}Z^*) = L(A):L(B)$. ■

3. POSOTONE FUNCTIONS

In this section we study posotone functions, and prove that the composition of two posotone functions is again a posotone function.

Let V and S be finite dimensional complex vector spaces. A function $\phi: P(V) \to P(S)$ a called a *posotone function* if there is a finite dimensional space W such that $S \subset W$, and a completely positive linear function $L: B(V) \to B(W)$ such that

$$\phi(A) = \mathbb{S}(L(A)) \tag{8}$$

for all operators $A \in P(V)$.

THEOREM 1. *Let ϕ be a posotone function. If A and B are positive operators on V, α is a non-negative scalar, and $\{E_n\}$ is a sequence of positive operators on V decreasing monotonically to* 0, *then*

$$\phi(A+B) \geqslant \phi(A) + \phi(B), \tag{9a}$$

$$\phi(\alpha A) = \alpha\phi(A), \tag{9b}$$

$$\phi(A:B) \leqslant \phi(A):\phi(B), \tag{9c}$$

$$\phi(A+E_n) \textit{ decreases monotonically to } \phi(A). \tag{9d}$$

Proof. For (9a), $\phi(A+B) = \mathbb{S}(L(A+B)) = \mathbb{S}(L(A)+L(B)) \geqslant \mathbb{S}(L(A)) + \mathbb{S}(L(B)) = \phi(A) + \phi(B)$, using (7h).

For (9b), by Lemma 2, $L(A:B) \leqslant L(A):L(B)$. Then by (7g) and (h), $\phi(A:B) = \mathbb{S}(L(A:B)) \leqslant \mathbb{S}(L(A):L(B)) = \mathbb{S}(L(A)) : \mathbb{S}(L(B)) = \phi(A):\phi(B)$.

The formula (9c) follows from (7e) directly, and the formula (9d) follows immediately from Lemma 2. ■

In proving the next two theorems, it will be convenient to partition a matrix for Z so that $Z^* = [Z^{1*}, Z^{2*}]$, with $\phi(A) = \mathbb{S}(L(A)) = Z^1 A^{(k)} Z^{1*} - Z^1 A^{(k)} Z^{2*} (Z^2 A^{(k)} Z^{2*})^+ Z^2 A^{(k)} Z^{1*}$.

THEOREM 2. *Let ϕ and ψ be posotone functions such that the composition $\phi \circ \psi$ is defined. Then $\phi \circ \psi$ is a posotone function.*

Proof. It is easy to prove directly that (9) holds for $\phi\circ\psi$; what is needed is the representation (8).

First, let us observe that if ϕ_1 and ϕ_2 are posotone functions, so is the direct sum $\phi_1\oplus\phi_2$. In fact, let

$$\phi_1(A)=\mathcal{S}\begin{bmatrix} Z^1 \\ Z^2 \end{bmatrix} A^{(k)}\begin{bmatrix} Z^{1*} Z^{2*} \end{bmatrix}$$

and

$$\phi_2(A)=\mathcal{S}\begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix} A^{(l)}\begin{bmatrix} Y^{1*} Y^{2*} \end{bmatrix}.$$

Then

$$\phi_1\oplus\phi_2(A)=\mathcal{S}\left\{\begin{bmatrix} Z^1 & 0 \\ 0 & Y^1 \\ Z^2 & 0 \\ 0 & Y^2 \end{bmatrix} A^{(k+l)}\begin{bmatrix} Z^{1*} & 0 & Z^{2*} & 0 \\ 0 & Y^{1*} & 0 & Y^{2*} \end{bmatrix}\right\},$$

where the shorted operator is to the subspace corresponding to the first two rows of the partitioned product matrix. Since $\phi_1\oplus\phi_2$ is of the form (8), $\phi_1\oplus\phi_2$ is a posotone function.

Next consider a congruence $\psi(A)=K\phi(A)K^*$, where ϕ is a posotone function. Then

$$\psi(A)=\mathcal{S}\left(\begin{bmatrix} KZ^1 \\ Z^2 \end{bmatrix} A^{(k)}\begin{bmatrix} Z^{1*}K^* & Z^{2*} \end{bmatrix}\right),$$

so that $\psi(A)$ is a posotone function.

It follows from the two previous cases that if ϕ_1 and ϕ_2 are posotone functions, so is $\phi_1+\phi_2$. In fact,

$$\phi_1(A)+\phi_2(A)=\begin{bmatrix} I & I \end{bmatrix}\begin{bmatrix} \phi_1(A) & 0 \\ 0 & \phi_2(A) \end{bmatrix}\begin{bmatrix} I \\ I \end{bmatrix}.$$

Finally, note that if $\phi(A)=\mathcal{S}(L(A))$ is a posotone function, then $\mathcal{T}(\mathcal{S}(L(A)))$ is again a posotone function, by (7f).

To complete the proof of the theorem, let $\phi(A)=\mathcal{S}(L(A))$ and $\psi(A)=\mathcal{T}(M(A))$. Then $\phi\circ\psi(A)=\mathcal{S}(L(\mathcal{T}(M(A))))$, and by the four cases considered above, $\phi\circ\psi$ is a posotone function. ∎

Although the class of posotone functions is closed under finite sums, an infinite series of posotone functions need not converge to a posotone function. For example, let

$$\phi\begin{bmatrix} a & b \\ b & c \end{bmatrix}=\sum_{j=1}^{\infty} a:\frac{c}{j^2}=\sum_{j=1}^{\infty}\frac{ac}{j^2a+c}. \tag{10}$$

Then the sum converges, but not to a posotone function, since the sum is not a rational function of a and c. No doubt a suitable infinite dimensional theory can be constructed in which the ϕ of (10) becomes posotone, but we shall not treat this question here.

4. DUALITY OF POSOTONE FUNCTIONS

In this section we will define the dual of a posotone function; the main theorem is that the dual function is again posotone. Our definition is chosen to agree with the electrical situation; the network theory consequences of our theorem are discussed in Sec. 5.

Let ϕ be a posotone function. The *dual function* $\phi^{\perp}$ is defined by the formula

$$\phi^{\perp}(A)=\left[\phi(A^{-1})\right]^{+} \tag{11}$$

whenever A is a positive invertible operator. In order to define $\phi^{\perp}$ for all positive operators A, we use the formula

$$\phi^{\perp}(A)=\lim_{\varepsilon\to 0^{+}}\phi^{\perp}(A+\varepsilon I) \tag{11$'$}$$

whenever A is a noninvertible positive operator.

THEOREM 3. *Let ϕ be a posotone function. Then $\phi^{\perp}$ is a posotone function.*

Proof. We will consider invertible operators A and prove that a representation of the form (8) holds for $\phi^{\perp}$. In view of (9d) it will then follow that $\phi^{\perp}$, as extended by (11′) to all positive operators, is a posotone function.

First we consider some special cases. Suppose that $\phi(A)=\mathcal{S}(A)$. Then if A is invertible,

$$A^{-1}=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}=\begin{bmatrix} \mathcal{S}(A)^{-1} & -\mathcal{S}(A)^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}\mathcal{S}(A)^{-1} & A_{22}^{-1}+A_{22}^{-1}A_{21}\mathcal{S}(A)^{-1}A_{12}A_{22}^{-1} \end{bmatrix}$$

as is well known. Define $\mathcal{S}^{\perp}(A)=A_{11}$. Since $\mathcal{S}^{\perp}(A^{-1})=\mathcal{S}(A)^{-1}$, it follows that $\mathcal{S}$ and $\mathcal{S}^{\perp}$ are duals; moreover,

$$\mathcal{S}^{\perp}(A)=[I \quad 0]A\begin{bmatrix} I^* \\ 0 \end{bmatrix}$$

is a posotone function.

More generally, suppose that ψ is a posotone function which maps invertible operators into invertible operators. Let $\phi(A)=\mathcal{S}(\psi(A))$; then $\phi^{\perp}(A)=\mathcal{S}^{\perp}(\psi^{\perp}(A))$. Similarly, if $\phi(A)=M\psi(A)M^*$, where M is an invertible matrix, then $\phi^{\perp}(A)=M^{*-1}\psi^{\perp}(A)M^{-1}$. In both of these cases, Theorem 2 assures that if $\psi^{\perp}$ is a posotone function, then so is $\phi^{\perp}$.

We can now prove the theorem for the case when the operator Z is surjective. In this case, we may write, in terms of a suitable basis, $Z=[H \; 0]$, where H is invertible. Then

$$L(A)=\mathcal{S}^{\perp}\begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix}^* A^{(k)}\begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix}, \tag{12}$$

so that

$$L^{\perp}(A)=\mathcal{S}\begin{bmatrix} H^{-1} & 0 \\ 0 & I \end{bmatrix}A^{(k)}\begin{bmatrix} H^{-1} & 0 \\ 0 & I \end{bmatrix}^*.$$

Then $\phi^{\perp}(A)=\mathcal{S}^{\perp}(L^{\perp}(A))$ is posotone by Theorem 2.

It remains to treat the case where the operator Z is not surjective. To do this we will redefine the function ϕ so that the previous analysis will apply. Let P be a projection operator from W onto ran (Z); then PZ is surjective, and for all $A\in\mathfrak{B}(V)$, $PZA=ZA$. If S is a subspace of ran(Z), then $\phi(A)=\mathcal{S}(ZA^{(k)}Z^*)=\mathcal{S}(PZA^{(k)}Z^*P^*)$; since PZ is surjective, (12) holds, with PZ used instead of Z. If S is not a subspace of ran(Z), let $T=S\cap$ ran (Z). From (7f) it follows that for any positive operator B defined on ran (Z), $\mathcal{S}(B)=\mathcal{T}(B)\oplus 0$, where 0 is the zero operator on the orthogonal complement $S\ominus T$. Now let $\psi(A)=\mathcal{T}(PZA^{(k)}Z^*P^*)$. Then for any $A\in\mathcal{P}(V)$, $\phi(A)=\psi(A)\oplus 0$, and thus $\phi^{\perp}(A)=\psi^{\perp}(A)\oplus 0$. But $\psi^{\perp}$ is a posotone function as shown above; by Theorem 2, so is $\psi^{\perp}\oplus 0$. ■

Given a specific representation $\phi(A)=\mathcal{S}(L(A))$ of the posotone function ϕ, one might wish to compute such a representation for ϕ. For the case of surjective Z, the change of basis implied in (12) can be avoided. Instead solve the linear equations $Z[Y \quad N]=[I \quad 0]$, choosing N to have the maximum possible number of independent columns. Then

$$\phi(A)=\mathcal{S}\left(\begin{bmatrix} Y^* \\ N^* \end{bmatrix} A^{(k)}[Y \quad N]\right), \tag{12'}$$

as may be verified from (12). [We are indebted to the referee for the suggestion that the use of (12) rather than (12′) would lead to a simpler proof of Theorem 3.]

5. POSOTONE FUNCTIONS AND ELECTRICAL NETWORKS

Our original motivation for studying posotone functions came from the study of electrical networks. An *n-port network* is a "black box" with $2n$ terminals divided into n pairs, called ports. At each port a voltage and current are measured, so that voltage and current become n dimensional vectors. The voltage vectors v and the current vector $\mathbf{i}$ are related by the equation $\mathrm{v}=Z\mathrm{i}$, where Z is a linear operator called the *impedance operator*. For *resistive networks* Z is a positive operator; we will consider only resistive networks here.

When two networks with impedance operators A and B are connected in series, the impedance of the new network is $A+B$; for parallel connections the new impedance is $A:B$ [3, 6]. If the ports of a network with impedance matrix A are connected to a transformer, the impedance of the new network is given by a congruence MAM^* [14]. If some of the ports of a network are shorted, the resulting impedance is given by a shorted operator [2]. Thus the various matrix operations considered in this study all have electrical interpretations. The various parts of Lemmas 1, 2 and 3 also have electrical interpretations.

The formula (8) which defines posotone functions can be interpreted as giving the impedance of a network consisting of k copies of A; each of which is connected to a transformer; then a series connection is performed; and finally some ports are shorted.

In electrical network theory the notion of duality arises from the interchange of current and voltage [9]. The admittance operator Y is defined by $\mathbf{i}=Y\mathrm{v}$; of course, if the impedance operator Z is invertible, then $Z^{-1}=Y$.

Given a posotone function ϕ, the dual function $\phi^{\perp}$ can always be realized by using gyrators [15]; the network content of Theorem 3 is that a realization without gyrators is always possible.

6. CONCLUSION

As mentioned in the previous section, the definition of the class of posotone functions was motivated by the connection of n-port networks using transformers. The restriction to completely positive functions is thus natural on physical grounds.

There are however, many other functions mapping $\mathcal{P}(V)$ to $\mathcal{P}(S)$ which satisfy (9). The function $\phi(A)=A^t$ of (3) clearly satisfies (9); we can show, however, that this ϕ is not posotone. To see this, suppose that $\phi(A)=A^t=\mathbb{S}(L(A))$. Then since ϕ is linear, $\mathbb{S}(L(A)+L(B))=\mathbb{S}(L(A))+\mathbb{S}(L(B))$ for all positive operators A and B. Then by (7h) there is a subspace T of W such that $S\cap T=0$ and such that $L(A)=L_1(A)+L_2(A)$ for all A; moreover, $L_1(A)=A^t$ and ran $(L_2(A))\subset T$. We may then use a construction similar to that which proved that ϕ was not completely positive, proving that $L^{(2)}$ is not positive, and thus $\phi(A)=A^t$ is not posotone (see Appendix).

There remains the question of determining necessary and sufficient conditions for a function to be posotone. Nishio and Ando have characterized the parallel sums and the shorted operator by conditions similar to (7) [19]; perhaps a similar treatment will succeed for posotone functions.

If the linear function L is merely positive; the formula $\phi(A)=\mathbb{S}(L(A))$ will still define a positive operator on S. It is still true that $\phi(A+B)\geqslant\phi(A)+\phi(B)$; we do not know if the dual inequality $\phi(A:B)\leqslant\phi(A):\phi(B)$ holds. If this second inequality does indeed hold, then an analogue of Theorem 2 can be proved without difficulty; Theorem 3 would presumably also hold in this wider context.

If the vector spaces are allowed to become infinite dimensional, then our definition of complete positivity no longer agrees with Stinespring's. A considerably different treatment seems necessary in order to prove analogues of Theorems 1, 2 and 3.

APPENDIX—PROOF THAT $\phi(A)=A^t$ IS NONPOSOTONE

Suppose that $\phi(A)=\mathbb{S}(L(A))$ for some positive linear function L. We consider a two dimensional space S, and assume that an orthonormal basis for W is given by extending an orthonormal basis for S. For a vector $x\in W$, we write

$$x=\left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \\ \vdots \\ x_m \end{array}\right],$$

where the first two coordinates correspond to S.

From the definition of ϕ, we have $\phi(A+B)=\phi(A)+\phi(B)$ for all positive operators A and B, so that

$$\mathbb{S}(L(A)+L(B))=\mathbb{S}(L(A))+\mathbb{S}(L(B)). \tag{13}$$

By (7h) the equality (13) will hold only if

$$\operatorname{ran}(L(A)-\phi(A)+L(B)-\phi(B))\cap S=0. \tag{14}$$

[Note: In order to simplify the notation we are identifying $\phi(A)$, the operator on S, with its extension $\phi(A)\oplus 0$, an operator on W.] Since $L(A)-\phi(A)$ is positive for all A, the condition (14) is equivalent to

$$\left[\operatorname{ran}(L(A)-\phi(A))+\operatorname{ran}(L(B)-\phi(B))\right]\cap S=0. \tag{14$'$}$$

Now let T be the subspace spanned by all the subspaces ran $L(A)-\phi(A)$ as A varies over all positive operators on S. Then $T\cap S=0$. If this were not the case, then there would be vectors a_i and operators A_i $(i=1,\dots,k)$ and a nonzero vector $s\in S$ such that $a_i\in$ ran $(L(A_i)-\phi(A_i))$ and $\sum_1^k a_i=s$. Now let $A=\sum_1^{k-1}A_i$ and $a=\sum_1^{k-1}a_i$. Then $a\in$ ran $(L(A)-\phi(A))$. But then $s=a+a_k\in[\operatorname{ran}(L(A)-\phi(A))+\operatorname{ran}(L(A_k)-\phi(A_k))]$, contradicting (14′).

For positive operators A, we may now write $L(A)=L_1(A)+L_2(A)$, with $L_1(A)$ and $L_2(A)$ positive operators such that ran $(L_1(A))\subset S$ and ran $(L_2(A))\subset T$. By Theorem 2 of [2], such a decomposition is unique; moreover, $L_1(A)=\mathbb{S}(A)$. In the present case this means that $L_1(A)=A^t$. Since the vector spaces are complex, L_1 and L_2 are determined by their actions on positive operators. [In fact, for any operator A, $L(A)=\frac{1}{2}L(A+A^*)-(i/2)$ $L(iA-iA^*)$, and $A+A^*$ and $iA-iA^*$ are Hermitian; moreover, any Hermitian operator is the difference of two positive operators.] Therefore $L_1(A)=A^t$ for all operators A, and ran $(L_2(A))\subset T$ for all operators A.

To prove that L is not completely positive, we will use a construction similar to that which proved that A^t was not completely positive.

Let P be the orthogonal projection from W onto S; then P maps $T^{\perp}$ onto S. To see this, observe that T could be extended to a subspace $\tilde{T}$ which is complementary to S. Then $\tilde{T}^{\perp}$ and S are both two dimensional, so that P will map $\tilde{T}^{\perp}$ onto S if and only if $P|\tilde{T}^{\perp}$ is injective. But if there is some nonzero vector $x \in \tilde{T}^{\perp} \cap S^{\perp}$, then $x \notin \tilde{T} + S$, which cannot happen, since $\tilde{T}$ and S were complementary. Thus P maps $\tilde{T}^{\perp}$ onto S; since $\tilde{T}^{\perp} \subset T^{\perp}$, the subspace $T^{\perp}$ will also be mapped onto S.

Let a and b be vectors in $T^{\perp}$ such that

$$Pa = \left[\begin{array}{c} 1\\ 0\\ \hline 0\\ \vdots\\ 0 \end{array}\right] \quad \text{and} \quad Pb = \left[\begin{array}{c} 0\\ 1\\ \hline 0\\ \vdots\\ 0 \end{array}\right]$$

so that

$$a = \left[\begin{array}{c} 1\\ 0\\ \hline a_3\\ \vdots\\ a_m \end{array}\right] \quad \text{and} \quad b = \left[\begin{array}{c} 0\\ 1\\ \hline b_3\\ \vdots\\ b_m \end{array}\right].$$

As before, let

$$M = \begin{bmatrix} 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 \end{bmatrix},$$

a positive operator on $S^{(2)}$. Then $L^{(2)}(M) = Q + R$, where

$$Q = \left[\begin{array}{cc|c|cc|c} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & & 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 & \multicolumn{2}{c|}{0} & 0\\ \hline 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & & 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & 0 & \multicolumn{2}{c|}{0} & 0 \end{array}\right] \quad \text{and} \quad R = \begin{bmatrix} R_1 & R_2\\ R_2^* & R_3 \end{bmatrix}.$$

The operator R is Hermitian by the definition of L; moreover, each R_i has range contained in T. Since R_1 is Hermitian with range contained in T, and $b \in T^{\perp}$, it follows that $R_1 b = 0$; similarly $R_3 a = 0$. Since $\mathrm{ran}(R_2) \subset T$, it follows that $T^{\perp} \subset \ker(R_2^*)$, so that $R_2^* b = 0$; similarly $R_2 a = 0$.

Now consider the inner product

$$\left(L^{(2)}(M)\begin{bmatrix} b \\ -a \end{bmatrix}, \begin{bmatrix} b \\ -a \end{bmatrix}\right) = \left(Q\begin{bmatrix} b \\ -a \end{bmatrix}, \begin{bmatrix} b \\ -a \end{bmatrix}\right) + \left(R\begin{bmatrix} b \\ -a \end{bmatrix}, \begin{bmatrix} b \\ -a \end{bmatrix}\right)$$

$$= -2 + 0 < 0.$$

It follows that L is not completely positive, as desired.

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